n-player threshold participation game 2

Following repeated episodes of insecurity in the neighborhood, the n residents of one block decide to get together to hire a police officer to patrol the street. Each resident can either contribute a fixed amount or not contribute.

In order to hire the police officer, at least m residents must contribute, where m is a parameter satisfying

$$2 \leq m \leq n$$

Each resident ranks the possible outcomes from best to worst as follows:

1. the public good is provided and she does not contribute
2. the public good is provided and she contributes
3. the public good is not provided and she does not contribute
4. the public good is not provided and she contributes

Formulate this situation as a game and find the Nash equilibria

Solution

Each resident $i = 1, \dots, n$ chooses one of two actions:

$$a_i \in \{C, N\}$$

where C means "contribute" and N means "not contribute"
Let

$$k(a) = \sum_{i=1}^{n} \mathbf{1}_{\{a_i = C\}}$$

denote the total number of contributors under the action profile $a = (a_1, \dots, a_n)$
The public good is provided if and only if

$$k(a) \geq m$$

To represent the preference ordering numerically, we may assign the following utilities to each resident i:

$$u_i(a) = \begin{cases} 4 & \text{if } k(a) \geq m \text{ and } a_i = N \\ 3 & \text{if } k(a) \geq m \text{ and } a_i = C \\ 2 & \text{if } k(a) < m \text{ and } a_i = N \\ 1 & \text{if } k(a) < m \text{ and } a_i = C \end{cases}$$

This utility function captures exactly the ranking given in the statement
We now characterize the pure-strategy Nash equilibria

Step 1: Profiles with fewer than m contributors

Suppose that

$$1 \leq k(a) < m$$

Then the public good is not provided. Every contributor receives utility 1
If one of those contributors deviates from C to N, the public good is still not provided, but now that resident receives utility 2
Thus every contributor would prefer to deviate
Therefore, no profile with

$$1 \leq k(a) < m$$

can be a Nash equilibrium

Step 2: Profiles with more than m contributors

Suppose that

$$k(a) > m$$

Then the public good is provided, and every contributor receives utility 3
If one contributor deviates from C to N, there are still at least m contributors, so the public good continues to be provided. The deviating resident then receives utility 4
Thus every contributor would prefer to deviate
Therefore, no profile with

$$k(a) > m$$

can be a Nash equilibrium

Step 3: Profiles with exactly m contributors

Suppose that

$$k(a) = m$$

Then the public good is provided

Each contributor receives utility 3. If one of them deviates to N, the number of contributors falls to $m - 1$, the public good is no longer provided, and that resident receives utility 2

So contributors do not want to deviate

Each non-contributor receives utility 4. If one of them deviates to C, the public good is still provided, but now that resident receives utility 3

So non-contributors do not want to deviate either

Hence, every profile with exactly m contributors is a Nash equilibrium

Step 4: The profile with no contributors

Finally, suppose that

$$k(a) = 0$$

Then the public good is not provided, and every resident receives utility 2

If one resident deviates unilaterally to C, then since

$$m \geq 2$$

the public good is still not provided, and the deviating resident receives utility 1

So nobody wants to deviate

Hence, the profile in which nobody contributes is also a Nash equilibrium

Conclusion

The pure-strategy Nash equilibria are:

1. the profile in which no resident contributes
2. every profile in which exactly m residents contribute

Therefore, the set of Nash equilibria consists of

$$\text{the all-}N\text{ profile and all action profiles with exactly } m \text{ contributors}$$

In particular, the total number of pure-strategy Nash equilibria is

$$1 + \binom{n}{m}$$